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# ON FUZZY IDEALS OF A RING

M.Mashinchi and M.M. Zahedi

*Department of Mathematics, Faculty of Science, University of Kerman, Kerman. Iran*

**Abstract**

The concepts of L-fuzzy ideal generated by a L-fuzzy subset, L-fuzzy prime and completely prime ideal where L is a complete lattice are considered and some results are proved.

## 1. Introduction

Zadeh [7] introduced the notion of a fuzzy subset of a set X as a function from X to [0,1]. Goguen in [1] replaced the lattice [0,1] by a complete lattice L and studied L-fuzzy subsets. Rosenfeld [4] used this concept and developed some results in fuzzy group theory. Wang-Jin Liu [5,6] studied fuzzy ideals of a ring. Mukherjee and Sen [2] studied fuzzy ideals further.

In this paper, for a complete lattice L, the concept of a L-fuzzy ideal generated by a L-fuzzy subset of a ring is considered and L-fuzzy ideal generated by a L-fuzzy point is characterized. Then by using this characterization and the concepts of L-fuzzy prime and completely prime ideals, it is proved that every L-fuzzy completelely prime ideal of a ring is a L-fuzzy prime ideal, and the converse is true whenever the ring is commutative.

## 2. Preliminaries

We fix $L=(L,\leq,\vee,\wedge)$ as a complete lattice with a least element 0 and greatest element 1. We write "sup" and "inf" for "$\vee$" and "$\wedge$", respectively. If $a,b\in L$ we write $b\geq a$ iff $a\leq b$. For a nonempty set X, let $F(X)=\{A|A$ is a L-fuzzy subset of $X\}$. Then for $A,B\in F(X)$, we write $A\subseteq B$ iff $A(x)\leq B(x)$ for all $x\in X$. By a L-fuzzy point $x_r$ of X; $x\in X$, $r\in L$, we mean $x_r\in F(X)$ defined by

$$x_r(y)=\begin{cases} r & \text{if } y=x \\ 0 & \text{otherwise,} \end{cases}$$

and we write $x_r\in X$. If $x_r\in X$ and $x_r\subseteq A\in F(X)$, then we write $x_r\in A$.

From now on R is a ring.

**Key words:** L-fuzzy ideal generated by a L-fuzzy subset, L-fuzzyprime, and completely prime ideal

**Definition 2.1.** Let $I\in F(R)$, then I is called a L-fuzzy ideal of R iff, for all $a,b\in R$

(i) $I(a-b)\geq \inf(I(a),I(b))$

(ii) $I(ab)\geq \sup(I(a),I(b))$.

We let I(R) be the set of all L-fuzzy ideals of R.

**Definition 2.2** [6, Proposition 3.4]. Let $I,J\in I(R)$, then $IJ\in I(R)$, is defined by

$$IJ(a)=\begin{cases} \sup \ \inf(I(a_1),\ldots,I(a_n),J(b_1)\ldots,J(b_n)) \\ \text{if } a=\sum_{i=1}^{n} a_ib_i;\ \text{for some } n\in\mathbb{N},\ a_i,b_i\in R \\ 0 \quad \text{if } a\neq\sum_{i=1}^{n} a_ib_i\ ;\ \text{for all } n\in\mathbb{N},\ a_i,b_i\in R, \end{cases}$$

**Lemma 2.3.** Let $a_t,b_s$ be two L-fuzzy points of R, then

$$a_tb_s=(ab)_{\inf(t,s)}.$$

**Proof.** follows directly from Definition 2.2.■

**Definition 2.4.** Let $A,B,A_\alpha\in F(R)$, where $\alpha$ is in the index set $\Lambda$. We define $A\cap B$, $\bigcap_{\alpha\in\Lambda}A_\alpha\in F(R)$ as follows,

(i) $A\cap B\ (r)=\inf\ (A(r),B(r))$ ; for all $r\in R$

(ii) $\bigcap_{\alpha\in\Lambda}A_\alpha\ (r)=\inf_{\alpha\in\Lambda} A_\alpha(r)$ ; for all $r\in R$.

**Definition 2.5.** Let $P\in I(R)$ be nonconstant. P is said to be a L−fuzzy completely prime ideal of R iff for any two L-fuzzy points $a_t,b_s$ of R,

$a_tb_s\in P$ implies either $a_t\in P$ or $b_s\in P$.

**Definition 2.6.** Let $P\in I(R)$ be nonconstant. P is said to be a L-fuzzy prime ideal of R iff for any $I,J\in I(R)$,

$IJ\subseteq P$ implies either $I\subseteq P$ or $J\subseteq P$.

### 3. Results

**Theorem 3.1.** Let $A \in F(R)$. Then the L-fuzzy subset $J = \bigcap_{A \subseteq I} I$, where $I \in I(R)$, is the smallest L-fuzzy ideal of R containing A. i.e. $A \subseteq J$ and for any $k \in I(R)$ such that $A \subseteq k$, then $J \subseteq k$.

**Proof.** The proof that $J \in I(R)$ partially follows from Proposition 3.1 of [4]. The rest of the proof is straightforward.■

**Definition 3.2** The L-fuzzy ideal J in Theorem 3.1 is called the L-fuzzy ideal generated by A, and is denoted by $<A>$.

**Theorem 3.3** For an arbitrary L-fuzzy point $a_t$ of R, $<a_t> = I$, where $I \in I(R)$ is defined by

$$\text{(i)} \quad I(r) = \begin{cases} t & \text{if } r = ab + ca + na + \sum_{i=1}^{m} a_i a b_i \text{ ; for some } b, c, a_i, b_i \in R,\ n \in \mathbb{Z},\ m \in \mathbb{N} \\ 0 & \text{otherwise,} \end{cases}$$

In particular,

$$\text{(ii)} \quad I(r) = \begin{cases} t & \text{if } r = ab + na \text{ ; for some } b \in R,\ n \in \mathbb{Z} \\ o & \text{otherwise} \end{cases}$$

if R is commutative, and

$$\text{(iii)} \quad I(r) = \begin{cases} t & \text{if } r = ab \text{ ; for some } b \in R \\ 0 & \text{otherwise.} \end{cases}$$

if R is commutative with identity 1.

**Proof** (i): First we show that $I \in I(R)$. Let $b, c \in R$. If $I(b) = 0$ or $I(c) = 0$, then $I(b-c) \geq 0 = \inf(I(b), I(c))$. Otherwise we have $I(b) = I(c) = t$ and

$$b = ad + ea + na + \sum_{i=1}^{m} a_i a b_i \text{ ; for some } d, e, a_i, b_i \in R,\ n \in \mathbb{Z},\ m \in \mathbb{N},$$

$$c = ad' + e'a + n'a + \sum_{j=1}^{m'} a'_j a b'_j \text{ ; for some } d', e', a'_j, b'_j \in R,\ n' \in \mathbb{Z},\ m' \in \mathbb{N}.$$

So we can write

$$b - c = ad'' + e''a + n''a + \sum_{i=1}^{m''} a''_i a b''_i \text{ ; for some } d'', e'', a''_i, b''_i \in R,\ n'' \in \mathbb{Z},\ m'' \in \mathbb{N}.$$

Therefore $I(b-c) = t = \inf(I(b), I(c))$. Hence

$$I(b-c) \geq \inf(I(b), I(c)) \text{ ; for all } b, c \in R.$$

Now for $b, c \in R$, if $I(c) = 0$ then $I(bc) \geq 0 = I(c)$ otherwise $I(c) = t$ and

$$c = ad + ea + na + \sum_{i=1}^{m} a_i a b_i \text{ ; for some } d, e, a_i, b_i \in R,\ n \in \mathbb{Z},\ m \in \mathbb{N}.$$

Thus

$$bc = bad + (be)a + (nb)a + \sum_{i=1}^{m} (ba_i) a b_i \text{ ; for some } d, e, a_i, b_i \in R,\ n \in \mathbb{Z},\ m \in \mathbb{N}.$$

So by definition of I, we have $I(bc) = t = I(c)$. Hence

$$I(bc) \geq I(c) \text{ ; for all } b, c \in R.$$

Similarly

$$I(bc) \geq I(b) \text{ ; for all } b, c \in R.$$

Thereby $I \in I(R)$. Now, since $I(a) = t$, so $a_t \in I$, i.e $a_t \in I$.

Next let $J \in I(R)$ and $a_t \in J$, we show that $I \subseteq J$. Consider $r \in R$, if $I(r) = 0$ then $J(r) \geq 0 = I(r)$ ; otherwise $I(r) = t$ and $r = ab + ca + na + \sum_{i=1}^{m} a_i a b_i$ ; for some $b, c, a_i, b_i \in R$, $n \in \mathbb{Z}$, $m \in \mathbb{N}$.

So

$$J(r) = J(ab + ca + na + \sum_{i=1}^{m} a_i a b_i)$$

$$\geq \inf(J(ab), J(ac), J(na), J(a_1 a b_1), \ldots, J(a_m a b_m)) \text{ ; by Definition 2.1(i)}$$

$$\geq \inf(J(a), J(a), \ldots, J(a)) \text{ ; which by Definition 2.1(ii) equals to } J(a)$$

$$\geq t \text{ ; since } a_t \in J$$

$$= I(r).$$

Therefore $J(r) \geq I(r)$ ; for all $r \in R$. Hence $I = <a_t>$, and (i) is proved. (ii), (iii) are special cases of (i).■

**Lemma 3.4.** If R is commutative, then $<a_t><b_s> = <a_t b_s>$ ; for all L-fuzzy points $a_t, b_s$ of R.

**Proof.** For arbitrary $r \in R$ ; let $\mathscr{A}_1$ = {all decomposition of r such that

$r=\sum_{i=1}^{n}(n_i a+ac_i)(m_i b+bd_i)$; for some $c_i,d_i \in R$, $n_i,m_i \in \mathbb{Z}$, $n \in \mathbb{N}\}$,

$\mathcal{A}_2=\{$all decomposition of r such that $r=\iota ab+(ab)c$; for some $\iota \in \mathbb{Z}, c \in R\}$.

Then it is easy to see that $\mathcal{A}_1=\mathcal{A}_2$. Now by using Definition 2.2, Theorem 3.3 (ii) and Lemma 2.3 we get

$$<a_t><b_s>(r)=\begin{cases} \inf(t,s) & \text{if } r\in\mathcal{A}_1 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

and

$$<a_t b_s>(r)=\begin{cases} \inf(t,s) & \text{if } r\in\mathcal{A}_2 \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

Thus by $\mathcal{A}_1=\mathcal{A}_2$ and (1), (2) the proof follows.■

**Theorem 3.5.** (i) Every L-fuzzy completely prime ideal of R is a L-fuzzy prime ideal of R. (ii) Conversely if R is commutative then a L-fuzzy prime ideal of R is a L-fuzzy completely prime ideal of R.

**Proof** (i): Let P be a L-fuzzy completely prime ideal and J,k any two L-fuzzy ideals of R such that $Jk \subseteq P$. We show that $J \subseteq P$ or $k \subseteq P$. Suppose $J \nsubseteq P$. So there exists $a \in R$ such that $J(a) \nleq P(a)$. Hence $a_{J(a)} \notin P$. Consider the L-fuzzy points $a_{J(a)} \in J$ and $b_{k(b)} \in k$, where b is an arbitrary element of R. Then for $r \in R$,

$$a_{J(a)}b_{k(b)}(r)=(ab)_{\inf(J(a),k(b))}(r)$$

$$=\begin{cases} \inf(J(a),k(b)) & \text{if } r=ab \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $a_{J(a)}b_{k(b)}(r)\leq Jk(r)\leq P(r)$. Hence $a_{J(a)}b_{k(b)} \in P$.

Thus $a_{J(a)} \in P$ or $b_{k(b)} \in P$. But $a_{J(a)} \notin P$, so $b_{k(b)} \in P$. Thereby $k(b)\leq P(b)$, and since b was arbitrary so $k \subseteq P$ and we are done.

(ii) Suppose P is a L-fuzzy prime ideal of R, and $a_t,b_s$ be two L-fuzzy points of R such that $a_t b_s \in P$. Then by Lemma 3.4 we have $<a_t><b_s>=<a_t b_s> \subseteq P$. So $<a_t> \subseteq P$ or $<b_s> \subseteq P$. Hence either $a_t \in P$ or $b_s \in P$, i.e. P is a L-fuzzy completely prime ideal of R.■

Added in proof. A recent paper of Mukherjee et al.[2] contains a special case of our Theorem 3.5, in which L is assumed to be [0,1]. Our results are independent and the proofs are different.

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